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Low dimensional cocommutative connected Hopf algebras

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Abstract

William M. Singer's theory of extensions of connected Hopf algebras is used to give a complete list of the cocommutative connected Hopf algebras over a field of positive characteristic p which have vector space dimension less than or equal to p^3. The theory shows that there are exactly two noncommutative nonprimitively generated Hopf algebras on the list, one of which is the Hopf algebra corresponding to the sub-Hopf algebra of the Steenrod algebra generated by P^1 and P^p. The commutative Hopf algebras are found using Borel's theorem and the primitively generated Hopf algebras using restricted Lie algebras.

0. Introduction

In this paper we study low dimensional cocommutative connected k-Hopf algebras for k a field of positive characteristic p using William M. Singer's theory of extensions of connected Hopf algebras [2]. Specifically, any finite dimensional cocommutative connected k-Hopf algebra occurs in a central extension $A \to C \to B$ where B is a cocommutative connected k-Hopf algebra of vector space dimension strictly less than that of C and A is polynomial on one generator truncated at height p or exterior on one generator. Singer describes a cohomology group $H^3(B, A)$ which classifies such extensions up to equivalence [2, proposition 5.1]. We use this group and induction on the dimension of C to give a complete list of the cocommutative connected k-Hopf algebras of vector space dimension less than or equal to p^3. A list of the commutative connected k-Hopf algebras of dimension less than or equal to p^3 could also be obtained by taking duals.

In practice it can be difficult to compute Singer's cohomology group, or even to calculate the Hopf algebra structure on C determined by an element in that group. Fortunately, we can avoid these computations for all but a few specific cases. When C is commutative and k is a perfect field, the algebra structure is determined by Borel's

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theorem [1, Theorem 7.11], and the possible coproducts are easily deduced when the dimension of C is small. If C is noncommutative but primitively generated, then it is the universal enveloping algebra of a nonabelian connected restricted Lie algebra [1, Theorem 6.11]. When the dimension of C is small, it is not difficult to give a list of these Lie algebras. We use Singer's theory for the remaining cases: C noncommutative and nonprimitively generated or C commutative and k not a perfect field. There are relatively few of these.

Since Singer's theory is not widely known, we give a brief summary in Section 1. A deep understanding of his results is not necessary for our purposes, but we will use his terminology and his definition of the group classifying extensions.

In Section 2 we study central extensions $A \to C \to B$ with A polynomial in one variable truncated at height p or exterior on one generator and with B and C cocommutative connected k-Hopf algebras. We are able to characterize those elements of $\boldsymbol{H}^3(B, A)$ which determine extensions with C commutative or with C primitively generated. We also construct a small piece of an exact sequence which is helpful in calculating $\boldsymbol{H}^3(B, A)$.

In Section 3 we apply these results to low dimensional cocommutative connected k-Hopf algebras. We show that there are only two pairs (A, B) which can give a C of dimension less than or equal to p^3 when C is noncommutative and nonprimitively generated. We calculate $\boldsymbol{H}^3(B, A)$ in these cases and show that there are only two such Hopf algebras. Borel's theorem, supplemented by the theory of Section 2 when k is not perfect, and restricted Lie algebras are used to complete the list.

1. Background material

In this section we establish our notation and give an overview of Singer's theory of extensions of connected Hopf algebras. We assume that the reader is familiar with the basic facts about Hopf algebras to be found in [1]. The reader will find a knowledge of the classification of extensions of groups helpful in understanding Singer's theory, although such a knowledge is not necessary for this results in this paper.

1.1. Notation

Let R be a commutative ring with unit. A *Hopf algebra* will mean a connected graded R-Hopf algebra in the sense of Milnor and Moore [1, Definition 4.1]. If C is such a Hopf algebra, we denote the product by μ_C, the coproduct by ψ_C, the unit by η_C, and the counit by ε_C. The category of cocommutative connected R-Hopf algebras has the tensor product as a product [1, p. 238] with projections

$$p_A : A \otimes B \xrightarrow{1 \otimes \varepsilon_B} A \otimes R \xrightarrow{\simeq} A$$

and

$$p_B : A \otimes B \xrightarrow{\varepsilon_A \otimes 1} R \otimes B \xrightarrow{\simeq} B.$$

There are also two inclusions

$$i_A : A \xrightarrow{\simeq} A \otimes R \xrightarrow{1 \otimes \eta_B} A \otimes B$$

and

$$i_B : B \xrightarrow{\simeq} R \otimes B \xrightarrow{\eta_A \otimes 1} A \otimes B.$$

when this notation is ambiguous we will use numbers as subscripts to indicate the factors in the tensor product to be used. Thus $p_{13} : B \otimes B \otimes A \to B \otimes A$ indicates projection on the first and third factors.

The maps which permute (with appropriate signs) the terms in a tensor product will be given by a list of numbers in parentheses. For instance

$$(1\,3\,2\,4) : A_1 \otimes A_2 \otimes A_3 \otimes A_4 \longrightarrow A_1 \otimes A_3 \otimes A_2 \otimes A_4.$$

Finally, $\langle x_1, \dots, x_n \rangle$ will be the free associative R-algebra, and $R\langle x_1, \dots, x_n \rangle$ the free R-module, generated by the set $\{x_1, \dots, x_n\}$.

1.2. Convolution products

The theory of extensions of groups makes use of the group structure on the set of group homomorphisms from G to H given by $(f_1 + f_2)(g) = f_1(g) + f_2(g)$, $f^{-1}(g) = -f(g)$, and $0(g) = 0$. With this product $\hom_{gp}(G, H)$ is an abelian group if H is commutative. The theory of extensions of connected Hopf algebras uses the group structure on the set of connected R-module morphisms from B to A given by the *convolution product* [1, Definition 8.1]: for $f, g \in \hom(B, A)$, $f * g = \mu_A(f \otimes g)\psi_B$. The unit is the map $\eta_A \varepsilon_B$ (the *trivial morphism*) and $\hom(B, A)$ is an abelian group when B is cocommutative and A is commutative. A connected R-module morphism is a R-module morphism which preserves the unit, i.e. $f(1) = 1$.

The properties of the convolution product are easily established and seem to be common knowledge, so we will use the basic properties without proof. Singer writes the convolution product additively, but we have chosen to write it multiplicatively to avoid confusion with the product coming from the module structure.

1.3. Singer's theory of extensions of connected Hopf algebras

Since Singer's theory does not appear to be widely known, we give a brief overview here. In actual fact we will only use his description of the group $H^3(B, A)$, which classifies extensions of B by A, and of how elements of that group correspond to extensions. His theory is essentially a self-dual version of the classical extension theory for groups.

An extension of Hopf algebras is a sequence of the form $A \to C \to B = C//A$, where A is left normal sub-Hopf algebra of C, together with a R-module splitting map $\gamma : B \to C$. Singer gives the following definition, which is equivalent by [1, Proposition 4.9].

Definition 1.1 (Singer [2, Definition 2.1]). An extension of a connected R-Hopf algebra B by a connected R-Hopf algebra A is a diagram of connected Hopf algebra morphisms

$$A \xrightarrow{\alpha} C \xrightarrow{\beta} B$$

together with a map $\lambda : C \to A \otimes B$ which is both an isomorphism of left A-modules and an isomorphism of right B-comodules.

C is a left A-module via the map α and the product on C and a right B-comodule via the coproduct on C and the map β.

Each extension determines an *action* σ_A of B on A and a *coaction* ρ_B of A on B [2, Definition 2.2 and Proposition 2.3]. As in the group theory case the action of b on a is computed by lifting b to C by the splitting map $\gamma : B \to C$, performing the Hopf algebra analogue of conjugation of $\alpha(a)$ by $\gamma(b)$, and pulling back to A. Thus $\alpha\sigma_A = (\gamma p_B) * (\alpha p_A) * (\gamma p_B)^{-1}$. The coaction is dual to this and can be thought of as "co-conjugation". The proofs of the following propositions are a useful exercise and are left to the reader.

Proposition 1.2. *If A is central in C, then the action $\sigma_A : B \otimes A \to A$ is the projection p_A (the* trivial action).

Proposition 1.3. *If C is cocommutative, then the coaction $\rho_B : B \to B \otimes A$ is the inclusion i_B (the* trivial coaction).

The module isomorphism $\lambda : C \to A \otimes B$ induces a product and a coproduct on $A \otimes B$. As in the group theory case the product can be described in terms of the action of B on A and a *twisting function* $\tau_A : B \otimes B \to A$ which measures the failure of the splitting map $\gamma : B \to C$ to preserve the product: $\alpha\tau_A = (\gamma p_1) * (\gamma p_2) * (\gamma\mu)^{-1}$. Dually, the coproduct can be described in terms of the coaction of A on B and a *cotwisting function* $\phi_B : B \to A \otimes A$. λ is an equivalence between the given extension and

$$A \xrightarrow{i_A} A \otimes B \xrightarrow{p_B} B$$

when $A \otimes B$ has this product and coproduct.

We will work exclusively with central extensions $A \to C \to B$ which have A commutative and both C and B cocommutative. In this case the action and coaction are trivial by Lemmas 1.2 and 1.3, and so the product and coproduct are given by

$$\mu(a \otimes b \otimes \bar{a} \otimes \bar{b}) = \sum \pm\, a\bar{a}\tau_A(b_{(1)}, \bar{b}_{(1)}) \otimes b_{(2)}\bar{b}_{(2)} \tag{1.1}$$

and

$$\psi(a \otimes \mathrm{b}) = \sum \pm\, a_{(1)}\phi_B^{(1)}b_{(1)}) \otimes b_{(2)} \otimes a_{(2)}\phi_B^{(2)}(b_{(1)}) \otimes b_{(3)}, \tag{1.2}$$

where $\phi_B(b) = \sum \phi_B^{(1)}(b) \otimes \phi_B^{(2)}(b)$ [2, Propositions 2.4 and 2.4*].

The classification of extensions thus reduces to the classification of pairs (τ_A, ϕ_B). Singer describes a cohomology theory associated to a pair of Hopf algebras (A, B) where A is commutative, B is cocommutative, A is a B-module, B is a A-module, and there are various conditions on the action and coaction. He calls such a pairs *abelian matched pairs* [2, Definition 3.1]. When the action and coaction are trivial, the extra conditions hold trivially and we have the *trivial abelian matched pair* (A, B) for any A commutative and B cocommutative.

The cohomology of a group G, $\boldsymbol{H}^*(G, A)$, can be computed using the normalized bar resolution of k as a module over the group ring of G and applying $\mathrm{hom}_{gp}(-, A)$ to get a chain complex. Singer makes this self-dual by taking an analogue of the normalized bar resolution of k as a B-module and an analogue of the normalized cobar resolution of k as an A-comodule and applying an appropriate hom to get a bicomplex [2, p. 11–12]:

$$\begin{tikzcd}
0 & {} & & \\
0 \arrow[u] \arrow[r] & \mathrm{hom}(B, A \otimes A) \arrow[u, "d_v"] \arrow[r, "d_h"] & \mathrm{hom}(B \otimes B, A \otimes A) & \\
0 \arrow[u] \arrow[r] & \mathrm{hom}(B, A) \arrow[u, "d_v"] \arrow[r, "d_h"] & \mathrm{hom}(B \otimes B, A) \arrow[u, "d_v"] \arrow[r, "d_h"] & {} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

His cohomology, $\boldsymbol{H}^*(B, A)$, comes from the total complex of this bicomplex. Thus the 3-cocycles are pairs (τ_A, ϕ_B) for $\tau_A : B \otimes B \to A$ and $\phi_B : B \to A \otimes \mathrm{A}$ satisfying five conditions [2, Proposition 4.1.5], given here for the trivial abelian matched pair:

The *unit condition* (from normalization in the bar resolution)

$$\tau_A i_1 = \eta_A \varepsilon_B = \tau_A i_2. \tag{1.3}$$

The *counit condition* (from normalization in the cobar resolution)

$$p_1 \phi_B = \eta_A \varepsilon_B = p_2 \phi_B. \tag{1.4}$$

The *associativity condition* (from $d_h \tau_A$ trivial)

$$(\tau_A p_{12}) * (\tau_A(\mu_B \otimes 1)) = (\tau_A p_{23}) * (\tau_A(1 \otimes \mu_B)). \tag{1.5}$$

The *coassociativity condition* (from $d_v \phi_B$ trivial)

$$((1 \otimes \psi_A)\phi_B) * (i_{23}\phi_B) = (\psi_A \otimes 1)\phi_B) * (i_{12}\phi_B). \tag{1.6}$$

The *Hopf condition* (from $d_v \tau_A = (d_h \phi_B)^{-1}$)

$$(\psi_A \tau_A) * (\phi_B \mu_B) = (\mu_{A \otimes A}(\phi_B \otimes \phi_B)) * ((\tau_A \otimes \tau_A)\psi_{B \otimes B}). \tag{1.7}$$

In other words, the 3-cocycles are pairs of twisting and cotwisting functions (τ_A, ϕ_B) defining a product and coproduct on $A \otimes \mathrm{B}$ by (1.1) and (1.2) which satisfy the conditions for a Hopf algebra.

Note that, when written additively, d_h and d_v resemble the differentials in the bar and cobar constructions, but they use the convolution product, and so are actually quite different.

The 3-coboundaries are pairs (τ_v, ϕ_v) defined by some $v: B \to A$ by [2, Proposition 4.1.6], given here for the trivial abelian matched pair:

$$\tau_v = (vp_1) * (v^{-1}\mu_B) * (vp_2), \tag{1.8}$$

$$\phi_v = (i_1 v^{-1}) * (\psi v) * (i_2 v^{-1}). \tag{1.9}$$

The 3-coboundaries give the extensions which are equivalent to the extension defined by a trivial twisting and cotwisting function (the *trivial extension*). Thus 3-cocycles differing by a 3-coboundary define equivalent extensions and $\boldsymbol{H}^3(B, A)$ classifies extensions of B by A up to equivalence [2, Proposition 5.1].

2. Extensions associated with cocommutative connected Hopf algebras

In this section A will denote a polynomial algebra on one generator truncated at height p or an exterior algebra on one generator. x will be the generator of A and $\mathrm{ht}(x)$ will be the height of x in A (either p or 2 respectively). Furthermore, B will denote a cocommutative connected k-Hopf algebra and all abelian matched pairs will be trivial. We begin by showing that every finite dimensional cocommutative connected k-Hopf algebra occurs in an extension $A \to C \to B$ corresponding to an element in $\boldsymbol{H}^3(B, A)$. Conversely, if $\mathrm{ht}(x) = p$, then every element of $\boldsymbol{H}^3(B, A)$ determines an extension with C cocommutative. If $\mathrm{ht}(x) = 2 < p$, then the elements that determine extensions with C cocommutative are precisely those which have a representative with a trivial cotwisting function.

Next we give necessary and sufficient conditions for an element in $\boldsymbol{H}^3(B, A)$ to determine an extension with C commutative or with C primitively generated. Lastly we construct an exact sequence which will be used to compute $\boldsymbol{H}^3(B, A)$.

2.1. Extensions with C cocommutative

Cocommutative connected k-Hopf algebras for k a field of positive characteristic p have many properties which are analogous to those of p-groups. The following lemma is analogous to the existence of nontrivial central elements in a p-group and shows that there is a simple type of extension which can be used to study cocommutative connected k-Hopf algebras.

Proposition 2.1. *If C is a nontrivial finite dimensional cocommutative connected k-Hopf algebra for k a field of positive characteristic p, then there is an extension $A \to C \to B$ determining an element in $\boldsymbol{H}^3(B, A)$ where B is a cocommutative connected k-Hopf algebra, A is either $k[x]/(x^p)$ with $|x|$ even if $p > 2$ or $\Lambda_k(x)$ with $|x|$ odd if $p > 2$, and (A, B) is the trivial abelian matched pair.*

Proof. Such a C always contains a primitive central element (see for example [3, Proposition 1.2]). Thus there is always a central sub-Hopf algebra A of the stated form and a central extension $A \to C \to B = C//A$. The action of B on A is trivial by Proposition 1.2 since A is central and the coaction of A on B is trivial by Proposition 1.3 since C is cocommutative. □

In general, not all elements in $\boldsymbol{H}^3(B, A)$ determine extensions with C cocommutative. However, for the extensions in Proposition 2.1 it is possible to completely characterize those elements which do. This provides a way to generate all the cocommutative connected k-Hopf algebras inductively using $\boldsymbol{H}^3(B, A)$.

Theorem 2.2. *Consider an extension $A \to C \to B$ corresponding to an element $u \in \boldsymbol{H}^3(B, A)$ where B is a cocommutative connected k-Hopf algebra and (A, B) is the trivial matched pair.*

(a) *If $A = k[x]/(x^p)$ with $|x|$ even if $p > 2$, then C is cocommutative.*

(b) *If $A = \Lambda_k(x)$ with $|x|$ odd and $p > 2$, then C is cocommutative if and only if there is a representative for u with trivial cotwisting function.*

Proof. The expression for the coproduct on $C \cong A \otimes \mathrm{B}$ (1.2) shows that C is cocommutative if and only if the cotwisting function for a representative of u is symmetric (in the graded sense). The theorem follows from the following lemma, which ensures the existence of a representative for u with a particularly simple cotwisting function.

Lemma 2.3. *Assume that B is a cocommutative connected k-Hopf algebra for k a field of positive characteristic p, that $A = k[x]/(x^p)$ or $\Lambda_k(x)$, and that (A, B) is the trivial abelian matched pair. If (τ_A, ϕ_B) is a cocycle representing an element of $\boldsymbol{H}^3(B, A)$, then*

(τ_A, ϕ_B) is cohomologous to a cocycle (τ'_A, ϕ'_B) with

$$\phi'_B(b) = \begin{cases} b & \text{if } |b| = 0, \\ \gamma(b)\left(\sum\limits_{\substack{r+s=\mathrm{ht}(x)\\ r,s>0}} \dfrac{1}{r!\,s!} x^r \otimes x^s \right) & \text{if } |b| = \mathrm{ht}(x)|x|, \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}$$

Here $\gamma : B_{\mathrm{ht}(x)|x|} \to k$ is some k-linear function. Furthermore, (τ'_A, ϕ'_B) can be chosen so that $\tau'_A = \tau_A$ in degrees less than $2|x|$.

Proof. $A \otimes A$ is concentrated in degrees which are multiples of $|x|$. Since $\phi_B : B \to A \otimes A$ is a connected module morphism which satisfies the counit condition (1.4), it must be trivial in degree less than $2|x|$. Now, if (τ''_A, ϕ''_B) is a cocycle such that ϕ''_B is trivial in degrees less than $l|x|$ for some $2 \le l < \mathrm{ht}(x)$, then the coassociativity condition (1.6) in degree $l|x|$ becomes $(1 \otimes \bar{\psi}_A)\phi''_B = (\bar{\psi}_A \otimes 1)\phi''_B$. A simple calculation shows that $\phi''_B = \gamma' \bar{\psi}_A(x^l)$ in that degree for some k-linear function $\gamma' : B_{l|x|} \to k$. Adding the coboundary (τ_ν, ϕ_ν) defined by

$$\nu(b) = \begin{cases} b & \text{if } |b| = 0, \\ -\gamma'(b)x^l & \text{if } |b| = l|x|, \\ 0 & \text{otherwise} \end{cases}$$

gives a cocycle (τ'_A, ϕ'_B) cohomologous to (τ''_A, ϕ''_B) with $\tau'_A = \tau''_A$ in degrees less than $2|x|$ (since $l \ge 2$) and ϕ'_B trivial in degrees less than $(l+1)|x|$. Inductively we obtain a cocycle (τ'_A, ϕ'_B) cohomologous to (τ_A, ϕ_B) with $\tau'_A = \tau_A$ in degrees less than $2|x|$ and ϕ'_B trivial in degrees less than $\mathrm{ht}(x)\,|x|$.

We use the coassociativity conditions (1.6) again to conclude that ϕ'_B satisfies (2.1) in degree $\mathrm{ht}(x)\,|x|$ and that it is trivial in all higher degrees. □

Proof of Theorem 2.2 (*continued*). Thus if $\mathrm{ht}(x) = p$, there is a representative for u with a cotwisting function which is given by (2.1). If $p > 2$, then $|x|$ is even, so the cotwisting function is symmetric and C is cocommutative. If $p = 2$, then signs are irrelavent, so the cotwisting function is symmetric and C cocommutative. On the other hand, if $\mathrm{ht}(x) = 2 < p$, then $|x|$ is odd and the cotwisting function has the form $\gamma \cdot (x \otimes x)$, which is symmetric only if γ is zero. Hence C is cocommutative if and only if u has a representative with a trivial cotwisting function. □

Note that the proof of Lemma 2.3 is essentially a calculation of $\mathrm{Cotor}^1_A(B, k)$, where B is a trivial A-comodule, using the cobar construction on A. The difference is that Lemma 2.3 uses the convolution product instead of the usual sum of module morphisms.

2.2. Extensions with C commutative

Consider a central extension $A \to C \to B$. As in the classification of extensions of groups, we think of the product in C as the products on A and B plus the value of τ_A. This suggests that C is commutative if and only if B is commutative and τ_A is symmetric (in the graded sense). Actually, we need to check the symmetry of τ_A only on the irreducibles of B, and when A is $k[x]/(x^p)$ or $\Lambda_k(x)$ we need to check only in degree $|x|$. The following theorem makes this precise.

Theorem 2.4. *Assume that B is a cocommutative connected k-Hopf algebra for k a field of positive characteristic p, that A is either $k[x]/(x^p)$ with $|x|$ even if $p > 2$ or $\Lambda_k(x)$ with $|x|$ odd if $p > 2$, and that (A, B) is the trivial abelian matched pair. If $u \in \boldsymbol{H}^3(B, A)$ determines an extension $A \to C \to B$ and (τ_A, ϕ_B) is a cocycle representing u, then C is commutative if and only if B is commutative and τ_A is trivial on $(QB \wedge QB)_{|x|}$.*

Proof. It is certainly a necessary condition that B be commutative, so we assume that this is the case. The expression for the product on $C \cong A \otimes B$ (1.1) shows that C is commutative if and only if the twisting function is symmetric for any representative of u.

Before proceeding, we need to show that τ_A gives a well defined map on $(QB \wedge QB)_{|x|}$. Since $\tau_A : B \otimes B \to A$, and A is trivial in degrees below $|x|$, the associativity condition (1.5) on $(IB \otimes IB \otimes IB)_{|x|}$ is $\tau_A(\mu_B \otimes 1) = \tau_A(1 \otimes \mu_B)$. Thus $\tau_A(bc, d) = \tau_A(b, cd)$ for $b \otimes c \otimes d \in (IB \otimes IB \otimes IB)_{|x|}$. Since B is commutative, this implies that $\tau_A(bc, d) = (-1)^{|bc||d|}\tau_A(d, bc)$, and therefore τ_A is well defined on $(QB \wedge QB)_{|x|}$.

If C is commutative, then τ_A is symmetric, and so is zero on $(QB \wedge QB)_{|x|}$. Conversely, if τ_A is zero on $(QB \wedge QB)_{|x|}$, we will show that τ_A is symmetric, and so C is commutative. Assume that τ_A is zero on $(QB \wedge QB)_{|x|}$. By the unit condition (1.3), it suffices to show that τ_A is symmetric on $IB \otimes IB$. Since A is concentrated in degrees $l|x|$ for $0 \le l < \mathrm{ht}(x)$, we only need to consider $(IB \otimes IB)_{l|x|}$. The case $l = 0$ is trivial. For $l = 1$, the initial assumption on τ_A and the fact that $\tau_A(bc, d) = (-1)^{|bc||d|}\tau_A(d, bc)$ for $b \otimes c \otimes d \in (IB \otimes IB \otimes IB)_{|x|}$ imply that τ_A is symmetric on all of $(IB \otimes IB)_{|x|}$.

To finish the proof, we argue by induction on l using the Hopf condition (1.7). To simplify the calculation, we assume that ϕ_B is trivial in degrees less than $\mathrm{ht}(x)|x|$. By Lemma 2.3, this can be done without loss of generality. Assume then that τ_A is symmetric in degrees less than $l|x|$ for some $1 < l < \mathrm{ht}(x)$ and take $b \otimes \bar{b} \in (IB \otimes IB)_{|x|}$. The Hopf condition on $b \otimes \bar{b}$ is

$$\psi_A \tau_A(b, \bar{b}) = \sum \pm \tau_A(b_{(1)}, \bar{b}_{(1)}) \otimes \tau_A(b_{(2)}, \bar{b}_{(2)}).$$

By assumption, the only terms on the right which are possibly nonsymmetric in b and $\bar{b}$ are $\tau_A(b, \bar{b}) \otimes 1$ and $1 \otimes \tau_A(b, \bar{b})$. Thus the Hopf condition implies that $\bar{\psi}_A \tau_A(b, \bar{b})$ is symmetric in b and $\bar{b}$. But $\tau_A(b, \bar{b})$ is a multiple of x^l in A, and since $l > 1$ it follows that $\tau_A(b, \bar{b})$ is symmetric. Hence τ_A is symmetric on $(IB \otimes IB)_{l|x|}$, and we are done. □

2.3. Extensions with C primitively generated

Intuitively, if C occurs in an extension $A \to C \to B$ with A primitively generated, then C is primitively generated if and only if there is a set of algebra generators in B which are primitive, and which remain primitive when lifted to C. The first part of this condition is that B be primitively generated and the second part is controlled by the behavior of the cotwisting function on primitive irreducibles. This is made precise in the following theorem.

Theorem 2.5. *Assume that B is a finite dimensional cocommutative connected k-Hopf algebra for k a field of positive characteristic p, that A is primitively generated, and that (A, B) is the trivial abelian matched pair. If $u \in H^3(B, A)$ determines an extension*

$$A \xrightarrow{\alpha} C \xrightarrow{\beta} B,$$

then C is primitively generated if and only if B is primitively generated and there is a representative for u with a cotwisting which is zero on the primitive irreducibles of B.

Proof. The following diagram is commutative with exact rows [1, Proposition 4.10].

$$\begin{array}{ccccccc} 0 \longrightarrow & PA & \longrightarrow & PC & \longrightarrow & PB & \\ & \downarrow & & \downarrow & & \downarrow & \\ & QA & \longrightarrow & QC & \longrightarrow & QB & \longrightarrow 0. \end{array} \tag{2.2}$$

Assume first that B is primitively generated and that (τ_A, ϕ_B) is a cocycle representing u with ϕ_B zero on the primitive irreducibles of B. Since B is primitively generated, we can choose a basis $\{b_i\}$ for QB such that b_i is primitive for all i. Since A is primitively generated, we can choose a basis $\{a_i\} \cup \{a_i'\}$ for A such that a_i and a_i' are primitive and the kernel of $QA \to QC$ is spanned by $\{a_i'\}$. Thus QC has a basis $\{a_i \otimes 1\} \cup \{1 \otimes b_i\}$. The formula for the coproduct on $C \cong A \otimes B(1.2)$ shows that $1 \otimes b_i$ and $a_i \otimes 1$ are primitive, and so C is primitively generated.

Conversely, if C is primitively generated, then so is B as a quotient of C. By [1, Proposition 6.16], $PC \cong PA \oplus PB$. Thus C has a vector space basis $\{\alpha(X)^I Y^J\}$ where $X = (x_1, \ldots, x_n)$ for $\{x_i\}$ a basis for PA, $Y = (y_1, \ldots, y_l)$ for y_i primitive in C and $\{\beta(y_i)\}$ a basis for PB, and $I = (i_i, \ldots, i_n)$ and $J = (j_1, \ldots, j_l)$ for $0 \le i_k < 2$ if $|x_k|$ odd, $0 \le i_k < p$ if $|x_k|$ even, $0 \le j_k < 2$ is $|y_k|$ odd, and $0 \le j_k < p$ if $|y_k|$ even. The map $\lambda: C \to A \otimes B$ given by $\lambda(\alpha(X)^I Y^J) = X^I \otimes \beta(Y)^J$ is clearly an isomorphism of left A-modules and right B-comodules, and thus determines a representative (τ_A, ϕ_B) for u [2, Proposition 2.11]. ϕ_B is given by Definition 2.2 and Proposition 2.3 of [2], and is easily seen to give zero on $\{\beta(y_i)\}$ (since y_i primitive and $p_A\lambda(y_i) = 0$), and hence on all the primitives of B. $\square$

2.4. An exact sequence

Lemma 2.3 says that the cotwisting function for an element in $H^3(B, A)$ is determined by a k-linear map $\gamma: B_{\mathrm{ht}(x)|x|} \to k$. The dual statement is that the twisting function is determined by a linear map $\gamma': (IB \otimes IB)_{|x|} \to k$, which is why the symmetry of τ_A only needs to be checked in degree $|x|$ in Theorem 2.4. Of course, the γ and γ' which come from representatives of elements in $H^3(B, A)$ are not arbitrary, and different representatives give different maps. The following theorem develops this idea.

Theorem 2.6. *Assume that B is a cocommutative connected k-Hopf algebra for k a field of positive characteristic p, that A is either $k[x]/(x^p)$ with $|x|$ even if $p > 2$* or *$\Lambda_k(x)$ with $|x|$ odd if $p > 2$, and that (A, B) is the trivial matched pair. There is an exact sequence of abelian groups*

$$H^{1,\mathrm{ht}(x)|x|}(B, k) \to H^3(B, A) \to H^{2,|x|}(B, k).$$

This is actually part of a longer exact sequence which will be described elsewhere. $H^{*,*}(B, k)$ is the ordinary cohomology of a Hopf algebra, i.e. $\mathrm{Ext}_B^{*,*}(k, k)$.

Proof of Theorem 2.6. $H^{*,*}(B, k)$ is the cohomology of the normalized bar construction on B:

$$\longrightarrow IB \otimes IB \otimes IB \xrightarrow{\bar{\mu} \otimes 1 - 1 \otimes \bar{\mu}} IB \otimes IB \xrightarrow{\bar{\mu}} IB \xrightarrow{0} k \longrightarrow 0,$$

where $IB^{\otimes n}$ is in external degree n. Thus

$$H^{1,\mathrm{ht}(x)|x|}(B, k) = \{\gamma: IB_{\mathrm{ht}(x)|x|} \to k \mid \gamma\bar{\mu} = 0\},$$

$$H^{2,|x|}(B, k) = \frac{\{\gamma': (IB \otimes \mathrm{IB})_{|x|} \to k \mid \gamma'(\bar{\mu} \otimes 1) = \gamma'(1 \otimes \bar{\mu})\}}{\{\lambda\bar{\mu} \mid \lambda: IB_{|x|} \to k\}}.$$

Consider $\Omega: H^{1,\mathrm{ht}(x)|x|}(B, k) \to H^3(B, A)$ induced by $\omega(\gamma) = (\tau'_A, \phi'_B)$ where τ'_A is trivial and ϕ'_B is given by (2.1). If γ is a cocycle, we claim that (τ'_A, ϕ'_B) is a cocycle. The unit, counit, and associativity conditions ((1.3), (1.4), and (1.5)) hold trivially, and the coassociativity condition (1.6) holds by the same calculation done in the proof of Lemma 2.3. Since τ'_A is trivial, the Hopf condition (1.7) is $\phi'_B\mu_B = \mu_{A \otimes A}(\phi'_B \otimes \phi'_B)$. The left-hand side is zero on $IB \otimes IB$ since $\gamma\bar{\mu} = 0$, while the right-hand side is zero on $IB \otimes IB$ since x has height $\mathrm{ht}(x)$. On the remaining part of $B \otimes B$ both sides give ϕ'_B. Note that ω also takes coboundaries to coboundaries since $\omega(0)$ has trivial twisting and cotwisting functions. Finally, if $\omega(\gamma_1) = (\tau_A, \phi_B)$ and $\omega(\gamma_2) = (\lambda'_A, \phi'_B)$, then the twisting function for $\omega(\gamma_1 + \gamma_2)$ is $\tau_A * \tau'_A$ since all are trivial. The cotwisting function for $\omega(\gamma_1 + \gamma_2)$ is $\phi_B * \phi'_B$ since both ϕ_B and ϕ'_B are trivial in degrees less than $\mathrm{ht}(x)\,|x|$, so in that degree $\phi_B * \phi'_B = \phi_B + \phi'_B$. Thus ω induces a group homomorphism on cohomology.

Next consider $\Theta : \boldsymbol{H}^3(B, A) \to \boldsymbol{H}^{2,|x|}(B, k)$ induced by $\theta(\tau_A, \phi_B) = \gamma'$, where $\tau_A(b, \bar{b}) = \gamma'(b, \bar{b})x$ for all $b \otimes \bar{b} \in (IB \otimes IB)_{|x|}$. If (τ_A, ϕ_B) is cocycle, then the associativity condition (1.5) in degree $|x|$ is $\tau_A(\mu_B \otimes 1) = \tau_A(1 \otimes \mu_B)$, which implies that γ' is a cocycle. If $(\tau_A, \phi_B) = (\tau_v, \phi_v)$ is the coboundary defined by $v : B \to A$, we consider $\lambda : IB_{|x|} \to k$ such that $v(b) = \lambda(b)x$ for $b \in IB_{|x|}$. The definition of τ_v (1.8) gives $\tau_v(b, \bar{b}) = -v(b\bar{b})$ for $b \otimes \bar{b} \in (IB \otimes IB)_{|x|}$ and so γ' is a coboundary since $\gamma' = -\lambda\bar{\mu}$. To see that θ is additive, simply note that $\tau_A * \tau'_A = \tau_A + \tau'_A$ in degree $|x|$.

It remains to show that the sequence is exact. Clearly $\theta\omega$ is zero, since $\omega(\gamma)$ has a trivial twisting function. Conversely, if (τ_A, τ_B) is a cocycle such that $\theta(\tau_A, \phi_B)$ is a coboundary, then $\tau_A(b, \bar{b}) = \gamma'(b\bar{b})x$ for $b \otimes \bar{b} \in (IB \otimes IB)_{|x|}$ for some $\gamma' : IB_{|x|} \to k$. Consider $v : B \to A$ given by

$$v(b) = \begin{cases} b & \text{if } |b| = 0, \\ \gamma'(b)x & \text{if } |b| = |x|, \\ 0 & \text{otherwise.} \end{cases}$$

If τ_v is the twisting function of the coboundary defined by v (1.8), then $(\tau_A * \tau_v)(b, \bar{b}) = \tau_A(b, \bar{b}) - v(b\bar{b}) = 0$ on $(IB \otimes IB)_{|x|}$. Thus (τ_A, ϕ_B) is cohomologous to a (τ'_A, ϕ'_B) with τ'_A trivial in degree $|x|$. By Lemma 2.3, we can assume that ϕ'_B is given by (2.1) for some γ. We claim that τ'_A is trivial in all degrees. Assume that τ'_A is trivial in degrees less than $l\,|x|$ for some $1 < l < \mathrm{ht}(x)$ and consider the Hopf condition (1.7) on $b \otimes \bar{b} \in (IB \otimes IB)_{|x|}$. Since ϕ'_B is trivial in this degree and below, the Hopf condition is $\bar{\psi}_A \tau'_A(b, \bar{b}) = 0$. But $\tau'_A(b, \bar{b})$ is a multiple of x^l, and so is primitive only if it is zero. Thus τ'_A is trivial and $(\tau'_A, \phi'_B) = \omega(\gamma)$. γ is a cocycle since the Hopf condition on $(IB \otimes IB)_{\mathrm{ht}(x)\,|x|}$ is $\phi'_B\mu_B = 0$ when the twisting function is trivial and the cotwisting function is given by (2.1). □

3. The classification of low dimensional cocommutative connected k-Hopf algebras

In this section we apply the results of Section 2 to obtain a list of the cocommutative connected k-Hopf algebras of vector space dimension less than or equal to p^3 when k is a field of positive characteristic p. We begin by using Borel's theorem to list the ones which are commutative when k is perfect, and restricted Lie algebras to list those which are noncommutative but primitively generated. We then use the results of Section 2 to show that there are only two which are noncommutative and nonprimitively generated. Finally, we indicate how the theory can be used to account for fields which are not perfect.

To simplify the terminology, we note that Proposition 2.1 implies that the vector space dimension of a finite dimensional cocommutative connected k-Hopf algebra is

$2^n p^m$. This is also a consequence of Borel's theorem [1, Proposition 7.11] if k is a perfect field. We call $n + m$ the *total rank* of the Hopf algebra.

3.1. The commutative cases

Theorem 3.1. *There are two commutative cocommutative connected k-Hopf algebras of total rank one over a field k of positive characteristic p:*

(a) $\Lambda_k(x)$ *with* $p > 2$ *and* $|x|$ *odd, and*

(b) $k[x]/(x^p)$ *for* $|x|$ *even if* $p > 2$.

All algebra generators are primitive. Type (a) *has dimension* 2 *and type* (b) *has dimension p.*

Proof. This is a direct consequence of [1, Proposition 7.8].

Theorem 3.2. *There are five commutative cocommutative connected k-Hopf algebras of total rank two over a perfect field k of positive characteristic p:*

(a) $\Lambda_k(y, z)$ *for* $p > 2$ *and* $|y|$ *and* $|z|$ *odd;*

(b) $\Lambda_k(y) \otimes k[z]/(z^p)$ *for* $p > 2$, $|y|$ *odd, and* $|z|$ *even;*

(c) $k[y]/(y^{p^2})$ *with* $|y|$ *even if* $p > 2$;

(d) $k[y, z]/(y^p, z^p)$ *with* $|y|$ *and* $|z|$ *even if* $p > 2$; *and*

(e) $k[y, z]/(y^p, z^p)$ *with* $|z| = p|y|$, $|y|$ *even if* $p > 2$, *and coproduct*

$$\bar{\psi}(z) = \sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} y^r \otimes y^s.$$

Unless otherwise stated, all algebra generators are primitive. Type (a) *has dimension* 4, *type* (b) *has dimension* $2p$, *and types* (c)–(e) *have dimension* p^2.

Proof. Proposition 7.21 of [1] implies that for $p > 2$, $\Lambda_k(x, y)$ with $|x|$ and $|y|$ odd is the only Hopf algebra of dimension 4. By the same reasoning, $\Lambda(x) \otimes k[y]/(y^p)$ with $|x|$ odd and $|y|$ even is the only Hopf algebra of dimension $2p$ for $p > 2$.

By Borel's theorem [1, Proposition 7.11], there are two possible algebra structures for a Hopf algebra of dimension $p^2 : k[y]/(y^{p^2})$ and $k[y, z]/(y^{p\cdot} z^p)$, where $|y|$ and $|z|$ are even if $p > 2$. For the first there is only one possible Hopf algebra structure, while for the second, we may assume that $|y| \leq |z|$, and so y is primitive. If z is primitive, the Hopf algebra structure is clear. If not, consider the dual. Since $QC^* \cong (PC)^*$, we know that C^* is monogenic and so $C^* = k[w]/(w^{p^2})$. A simple calculation shows that the coproduct on z has the form stated in the Hopf algebra of type (e). □

Theorem 3.3. *There are* 13 *commutative cocommutative connected k-Hopf algebras of total rank three over a perfect field k of positive characteristic p:*

type (a) ($p > 2$, *dimension* 8):

$$\Lambda(x, y, z), \quad |x|, |y|, \textit{ and } |z| \textit{ odd;}$$

type (b) ($p > 2$, *dimension* $4p$):

$$\Lambda(x, y) \otimes k[z]/(z^p), \quad |x| \text{ and } |y| \text{ odd}, |z| \text{ even};$$

type (c) ($p > 2$, *dimension* $2p^2$):

$$\Lambda(x) \otimes k[y]/(y^{p^2}), \quad |x| \text{ odd and } |y| \text{ even};$$

type (d) ($p > 2$, *dimension* $2p^2$):

$$\Lambda(x) \otimes k[y, z]/(y^p, z^p), \quad |x| \text{ odd}, |y| \text{ and } |z| \text{ even};$$

type (e) ($p > 2$, *dimension* $2p^2$):

$$\Lambda(x) \otimes k[y, z]/(y^p, z^p), \quad |x| \text{ odd}, |y| \text{ even}, |z| = p|y|, \text{ and}$$

$$\bar{\psi}(z) = \sum_{\substack{r+s=p\\ r,s>0}} \frac{1}{r!s!} y^r \otimes y^s;$$

type (f) (*dimension* p^3):

$$k[x]/(x^{p^3}), \quad |x| \text{ even if } p > 2;$$

type (g) (*dimension* p^3):

$$k[x, y]/(x^p, y^{p^2}) \quad |x| \text{ and } |y| \text{ even} \quad \text{if } p > 2,$$

type (g-1):

$$\bar{\psi}(x) = 0, \quad \bar{\psi}(y) = 0,$$

type (g-2):

$$|y| = p|x|, \quad \bar{\psi}(x) = 0, \quad \bar{\psi}(y) = \sum_{\substack{r+s=p\\ r,s>0}} \frac{1}{r!s!} x^r \otimes x^s,$$

type (g-3):

$$|x| = p|y|, \quad \bar{\psi}(x) = \sum_{\substack{r+s=p\\ r,s>0}} \frac{1}{r!s!} y^r \otimes y^s, \quad \bar{\psi}(y) = 0,$$

type (g-4):

$$|x| = p^2|y|, \quad \bar{\psi}(x) = \sum_{\substack{r+s=p\\ r,s>0}} \frac{1}{r!s!} y^{rp} \otimes y^{sp}, \quad \bar{\psi}(y) = 0;$$

type (h) (*dimension* p^3):

$$k[x, y, z]/(x^p, y^p, z^p) \quad |x|, |y|, \text{ and } |z| \text{ even} \quad \text{if } p > 2,$$

type (h-1):

$$\bar{\psi}(x) = 0, \quad \bar{\psi}(y) = 0, \quad \bar{\psi}(z) = 0,$$

type (h-2):

$$|z| = p|y|, \quad \bar{\psi}(x) = 0, \quad \bar{\psi}(y) = 0, \quad \bar{\psi}(z) = \sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} y^r \otimes y^s,$$

type (h-3):

$$|y| = p|x|, \quad |z| = p^2|x|, \quad \bar{\psi}(x) = 0,$$

$$\bar{\psi}(y) = \sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} x^r \otimes x^s, \qquad \bar{\psi}(z) = \sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} y^r \otimes y^s - \bar{\psi}(y)\psi(y)^{p-1}.$$

Unless otherwise mentioned, all algebra generators are primitive.

Proof. For $p > 2$, the Hopf algebras with dimensions other than p^3 (types (a)–(e)) are given by Proposition 7.21 of [1] and Theorem 3.2. The algebra structures of the remaining Hopf algebras are given by Borel's theorem [1, Theorem 7.11]. There is only one possible Hopf algebra structure on type (f). The Hopf algebras of type (g-1) and (h-1) are the primitively generated cases, and we proceed to classify the nonprimitively generated commutative Hopf algebras with QC either two or three dimensional.

Assume that C is nonprimitively generated and $C = k[x, y]/(x^p, y^{p^2})$ (thus $|y| \neq |x|$). If $|x| < |y|$, then x is primitive, but $[y] \in QC$ cannot be represented by a primitive element. A calculation with the coassociativity condition on ψ_C which is very similar to the calculation done for ϕ_B in Lemma 2.3 shows that type (g-2) is the only possibility. If $|y| < |x|$, then y is primitive and $[x] \in QC$ cannot be represented by a primitive element. If $|x| = p|y|$, then coassociativity gives type (g-3). On the other hand, if $|x| \neq p|y|$, we consider the dual of C. $PC^* = \langle \bar{x}, \bar{y} \rangle$ and $QC^* = \langle \bar{y}, \bar{z} \rangle$, where $\bar{x}$ is dual to x, $\bar{y}$ is dual to y, and $\bar{z}$ is dual to y^p. Since $\bar{y}^p$ must be primitive but is linearly independent of $\bar{x}$ and $\bar{y}$, it must be zero, and this together with the relative degrees of $\bar{y}$ and $\bar{z}$ shows that C^* is of type (g-2). A simple calculation shows that the dual of type (g-2) is of type (g-4).

Lastly, assume that C is nonprimitively generated and $C = k[x, y, z]/(x^p, y^p, z^p)$. If x and y are primitive but z is not, then $QC^* = \langle x^*, y^* \rangle$ and $PC^* = \langle x^*, y^*, z^* \rangle$, and so C^* is primitively generated with two algebra generators. Direct calculation shows that the dual type of type (g-1) is of type (h-2). If x is primitive by y and z are not, then $QC^* = \langle \bar{x} \rangle$, so C^* is of type (f). Again, direct calculation shows that the dual of type (f) is of type (h-3). $\square$

3.2. The noncommutative primitively generated cases

Lemma 3.4. *There are no noncommutative primitively generated k-Hopf algebras of total rank one for k a field of positive characteristic p.*

Proof. If C is such a Hopf algebra, then [1, Theorem 6.11] implies that C is the universal enveloping algebra of a nonabelian connected restricted Lie algebra of dimension one. But a one dimensional Lie algebra must be abelian, so this is impossible. $\square$

Theorem 3.5. *There is only one noncommutative primitively generated connected Hopf algebra of total rank two over a field k of positive characteristic p:*

$$\langle y\rangle/(y^{2p}),$$

where the brackets denote the free associative k-algebra generated by y, $|y|$ is odd, and $p > 2$.

Proof. The noncommutative primitively generated connected Hopf algebras of total rank two are the enveloping algebras of the nonabelian connected restricted Lie algebras of dimension two [1, Theorem 6.11]. Assume that L is such a restricted Lie algebra. The first case to consider is $p > 2$ and $L_{\text{odd}} = k\langle y, z\rangle$. Since the commutator of odd degree elements is in even degrees, this Lie algebra must be abelian. The next case is $p > 2$, $L_{\text{odd}} = k\langle y\rangle$, and $L_{\text{even}} = k\langle z\rangle$, and here there is only one possible nontrivial commutator, which by proper choice of generators is $[y, y] = z$. The restriction map must be trivial by degree considerations, and this gives the Hopf algebra listed in the statement of the theorem. The last case is $p > 2$, $L_{\text{odd}} = 0$, and $L_{\text{even}} = k\langle y, z\rangle$ or $p = 2$ and $L = k\langle y, z\rangle$. Here the commutator is forced to be zero and so L is abelian. $\square$

Theorem 3.6. *There are* 12 *noncommutative primitively generated connected Hopf algebras of total rank three over a field k of positive characteristic p:*
type (a) ($p > 2$, *dimension* $4p$):

$$\langle y\rangle/(y^{2p}) \otimes \Lambda_k(z), \quad |y| \textit{ and } |z| \textit{ odd};$$

type (b) ($p > 2$, *dimension* $4p$):

$$\langle y, z\rangle/(y^{2p}, z^2 - ly^2, [y, z]), \quad l \textit{ nonzero in } k, \quad |z| = |y|, \quad |y| \textit{ odd};$$

type (c) ($p > 2$, *dimension* $4p$):

$$\langle y, z\rangle/(y^{2p}, z^2, [y, z] - y^2), \quad |z| = |y|, \quad |y| \textit{ odd};$$

type (d) ($p > 2$, *dimension* $4p$):

$$\langle y, z\rangle/(y^{2p}, z^2 - ly^2, [y, z] - y^2), \quad l \textit{ nonzero in } k, \quad |z| = |y|, \quad |y| \textit{ odd};$$

type (e) ($p > 2$, *dimension* $4p$):

$$\langle x, y, z\rangle/(x^p, y^2, z^2, [x, y], [x, z], [y, z] - x), \quad |x| = |y| + |z|, \quad |y| \textit{ and } |z| \textit{ odd};$$

type (f) ($p > 2$, *dimension* $4p$):

$$\langle x, y, z\rangle/(x^2, y^2, z^p, [x, y], [x, z], [y, z] - x), \quad |x| = |y| + |z|$$

$$|y| \textit{ odd and } |z| \textit{ even};$$

type (g) ($p > 2$, *dimension* $2p^2$):

$$\langle y\rangle/(y^{2p^2}), \quad |y| \text{ odd};$$

type (h) ($p > 2$, *dimension* $2p^2$):

$$\langle y\rangle/(y^{2p}) \otimes k[z]/(z^p) \quad |y| \text{ odd and } |z| \text{ even};$$

type (i) ($p > 2$, *dimension* $2p^2$):

$$\langle y, z\rangle/(y^{2p}, z^p - ly^2, [y, z]) \quad l \text{ nonzero in } k,\ p|z| = 2|y|,\ |y| \text{ odd and } |z| \text{ even};$$

type (j) (*dimension* p^3):

$$\langle x, y, z\rangle/(x^p, y^p, z^p, [x, y], [x, z], [y, z] - x), \quad |x| = |y| + |z|,\ |y| \text{ and } |z| \text{ even} \quad \text{if } p > 2;$$

type (k) (*dimension* p^3):

$$\langle y, z\rangle/(y^{p^2}, z^p, [y, z] - y^p) \quad |z| = (p - 1)|y|,\ |y| \text{ even if } p > 2.$$

type (l) ($p = 2$, *dimension* 8):

$$\langle y, z\rangle/(y^4, z^4, y^2 - z^2, [y, z] - y^2) \quad |z| = |y|.$$

Again, the brackets denote the free associative k-algebra generated by the given elements. If $p = 2$ and $|y| = |z|$, then types (j) *and* (k) *coincide. Depending on the field k, some of the Hopf algebras in case* (b) *or case* (i) *may coincide for certain values of l.*

Proof. As before, the noncommutative primitively generated connected Hopf algebras of total rank three are the enveloping algebras of the nonabelian connected restricted Lie algebras of dimension three [1, Theorem 6.11]. Assume that L is such a restricted Lie algebra. The first case to consider is $p > 2$ and $L_{\text{odd}} = k\langle x, y, z\rangle$. In this case it is impossible for L to be nonabelian since the commutator of odd degree elements lies in even degree.

The next case is $p > 2$, $L_{\text{odd}} = k\langle y, z\rangle$, and $L_{\text{even}} = k\langle x\rangle$. Degree and scaling considerations give the following combinations of nonzero commutators to be considered:

(1) $[y, y] = x$ ($|x| = 2|y|$);
(2) $[y, y] = x$ and $[z, z] = lx$ for l nonzero in k ($|x| = 2|y|$ *and* $|z| = |y|$);
(3) $[y, y] = x$ and $[y, z] = x$ ($|x| = 2|y|$ and $|z| = |y|$);
(4) $[y, y] = x$, $[z, z] = lx$, and $[y, z] = x$ for l nonzero in k ($|x| = 2|y|$ and $|z| = |y|$);
(5) $[y, z] = x$ ($|x| = |y| + |z|$); and
(6) $[x, z] = y$ ($|y| = |x| + |z|$).

Because of the degrees the restriction map on x must be trivial. This gives the Hopf algebras of types (a), (b), (c), (d), (e), and (f), where we have relabeled the generators in (f) so that x is the central element.

The third case is $p > 2$, $L_{\text{odd}} = k\langle x\rangle$, and $L_{\text{even}} = k\langle y, z\rangle$. In this case the degrees of x, y, and z allow only one nontrivial commutator, and without loss of generality we have $[x, x] = y$. Considering the restriction maps gives three cases:

(1) $[x, x] = y$, $\xi(y) = z$, and $\xi(z) = 0$ ($|y| = 2|x|$ and $|z| = p|y|$);
(2) $[x, x] = y$, $\xi(y) = 0$, and $\xi(z) = 0$ ($|y| = 2|x|$).
(3) $[x, x] = y$, $\xi(y) = 0$, and $\xi(z) = ly$ for l nonzero in k ($|y| = 2|x|$, and $p|z| = 2|x|$).

These give the Hopf algebras of type (g), (h) and (i) respectively, and as before, we have relabeled the generators.

The fourth and last case is for $p > 2$, L_{odd} trivial, and $L_{\text{even}} = k\langle x, y, z\rangle$ or $p = 2$ and $L = k\langle x, y, z\rangle$. Without loss of generality, we can assume that $[y, z] = x$, and the degrees then imply that $[x, y] = [x, z] = 0$ and $\xi(x) = 0$. Again, degree and scaling considerations give the following nonzero commutators and restriction maps:

(1) $[y, z] = x$ ($|x| = |y| + |z|$);
(2) $[y, z] = x$ and $\xi(y) = x$ ($|x| = p|y|$ and $|z| = (p-1)|y|$);
(3) $[y, z] = x$, $\xi(y) = x$, and $\xi(z) = x$ ($p = 2$, $|x| = 2|y|$, and $|z| = |y|$).

These give the Hopf algebras of types (j), (k), and (l) respectively.

To see that type (j) and (k) are isomorphic when $p = 2$ and $|y| = |z|$, consider the map from type (k) with generators y and z to type (j) with generators $\bar{x}$, $\bar{y}$, and $\bar{z}$ given by $\gamma(y) = \bar{y} + \bar{z}$ and $\gamma(z) = \bar{y}$. It is easily checked that this is an isomorphism of Hopf algebras. □

3.3. *The noncommutative nonprimitively generated cases*

Lemma 3.7. *There are no noncommutative nonprimitively generated cocommutative connected k-Hopf algebras of total rank one over a field k of positive characteristic p.*

Proof. By Lemma 2.1, this is clear. □

Theorem 3.8. *There are no noncommutative nonprimitively generated cocommutative connected k-Hopf algebras of total rank two over a field k of positive characteristic p.*

Proof. By Lemma 2.1, a noncommutative nonprimitively generated cocommutative Hopf algebra C of total rank two occurs in an extension $A \to C \to B$ with $A = k[x]/(x^p)$ or $\Lambda_k(x)$ and B of rank one. Since all such B are commutative, Theorem 2.4 implies that $QB \wedge QB$ must be nontrivial in degree $|x|$. This in turn implies that $B = \Lambda_k(y)$ with $|y|$ odd, $p > 2$, and $|x| = 2|y|$. Thus $|x|$ is even and so $A = k[x]/(x^p)$. Now, since B is primitively generated, Theorem 2.5 and Lemma 2.3 imply that $QB_{2p|y|}$ is nontrivial, but this is impossible since QB is concentrated in degree $|y|$. □

Theorem 3.9. *There are two noncommutative nonprimitively generated cocommutative connected k-Hopf algebras of total rank three over a perfect field k of positive characteristic p.*

Type (a) ($p > 2$, *dimension* $2p^2$):

$$\langle y\rangle/(y^{2p}) \otimes k[z]/(z^p),$$

$$|z| = 2p|y|, \quad |y| \text{ odd},$$

$$\bar{\psi}(z) = \sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} y^{2r} \otimes y^{2s};$$

Type (b) (*dimension* p^3):

$$\langle x, y, z\rangle/(x^p, y^p, z^p - x^{p-1}y, [x, y]\,[x, z], [y, z] - x),$$

$$|x| = (p+1)|y|, \quad |z| = p|y|, \quad |y| \text{ even} \quad \text{if } p > 2,$$

$$\bar{\psi}(x) = \bar{\psi}(y) = 0, \quad \bar{\psi}(z) = \sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} y^r \otimes y^s.$$

The brackets denote the free associative k-algebra generated by the given elements, and generators are primitive unless otherwise noted.

Proof. By Lemma 2.1 a noncommutative nonprimitively generated cocommutative connected k-Hopf algebra C of rank three occurs in an extension $A \to C \to B$ with $A = k[x]/(x^p)$ or $\Lambda_k(x)$ and B of rank two. Thus there are six cases to consider.

Case 1: $B = \langle y\rangle/(y^{p^2})$ *for* $p > 2$ *and* $|y|$ *odd.* B is primitively generated, so by Theorem 2.5 and Lemma 2.3 we must have $QB_{\mathrm{ht}(x)|x|}$ nontrivial. But this is impossible since $\mathrm{ht}(x)|x|$ is always even and QB is concentrated in odd degrees.

Case 2: $B = \Lambda_k(y, z)$ *for* $p > 2$ *and* $|y|$, $|z|$ *odd.* As in the previous case $QB_{\mathrm{ht}(x)|x|}$ must be nontrivial. But QB is concentrated in odd degrees, so this is impossible.

Case 3: $B = \Lambda_k(y) \otimes k[z]/(z^p)$ *for* $p > 2$, $|y|$ *odd, and* $|z|$ *even.* Again, $QB_{\mathrm{ht}(x)|x|}$ must be nontrivial, and so $|z| = \mathrm{ht}(x)|x|$. Since B is commutative, Theorem 2.4 implies that $(QB \wedge QB)_{|x|} = k\langle y \otimes y, y \otimes z - z \otimes y\rangle_{|x|}$ is nontrivial. Degree considerations imply that $|x| = 2|y|$, and so $A = k[x]/(x^p)$ and $|z| = 2p|y|$.

The exact sequence from Theorem 2.6 and Lemma 2.3 in this case show that elements of $H^3(B, A)$ are represented by (τ_A, ϕ_B) with τ_A determined by $\tau_A(y, y) = ax$ and ϕ_B trivial except for

$$\phi_B(z) = b\left(\sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} x^r \otimes x^s\right).$$

Elements giving C noncommutative and nonprimitively generated have a and b nonzero, and by rescaling x and z we can assume $a = b = 1$. (The extensions for the different (a, b) with a and b nonzero are distinct, but the C they determine are isomorphic).

By (2.2), the algebra generators C are $\bar{y} = 1 \otimes y$, $\bar{z} = 1 \otimes z$, and $\bar{x} = x \otimes 1$ if the later is irreducible. A simple application of the formula for the product (1.1) shows that $[\bar{x}, \bar{y}] = [\bar{x}, \bar{z}] = [\bar{y}, \bar{z}] = 0$, the last since $\tau_A(y, z)$ is in dimension $(2p + 1)|y| > p|x|$, and so is zero. Since $\tau_A(y, y) = x$, we have $\bar{y}^2 = \bar{x}$ and so $\{\bar{y}^i \bar{z}^j\}$ spans C.

From Singer's formula for the coproduct (1.2) it is easy to see that $\bar{y}$ is primitive and

$$\bar{\psi}(\bar{z}) = \sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} \bar{x}^r \otimes \bar{x}^s.$$

Since $\bar{y}$ is primitive, it generates a sub-Hopf algebra, and so the height of $\bar{y}$ must be $2p$. As a consequence $\bar{z}^p$ must be a power of $\bar{y}$. Degree considerations show that $\bar{z}^p = 0$ and this gives the Hopf algebra of type (a).

Case 4: $B = k[y]/(y^{p^2})$ *for* $|y|$ *even if* $p > 2$. B is commutative and $QB \wedge QB$ is trivial, which is impossible by Theorem 2.4.

Case 5: $B = k[y, z]/(y^p, z^p)$ for $|y|$, $|z|$ *even if* $p > 2$. B is commutative and $QB \wedge QB = k\langle y \otimes z - z \otimes y \rangle$, so by Theorem 2.4, we have $|x| = |y| + |z|$ and $A = k[x]/(x^p)$. B is primitively generated, so by Theorem 2.5 and Lemma 2.3 we must have $QB_{p|x|}$ nontrivial. But this is impossible since $|y|, |z| < |x|$.

Case 6: $B = k[y, z]/(y^p, z^p)$ *with*

$$\bar{\psi}(z) = \sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} y^r \otimes y^s$$

and $|y|$, $|z|$ *even if* $p > 2$. As in Case 5 we have $|x| = |y| + |z|$, but since B is not primitively generated, there is no problem with $QB_{p|x|}$ being zero. Thus $A = k[x]/(x^p)$ since $|x|$ is even if $p > 2$.

As in Case 3, the exact sequence from Theorem 2.6 and Lemma 2.3 imply the elements of $\boldsymbol{H}^3(A, B)$ have representatives (τ_A, ϕ_B) with ϕ_B trivial and τ_A determined by $\tau_A(y, z) - \tau_A(z, y) = ax$. Elements giving C noncommutative have a nonzero, and all will be nonprimitively generated. Scaling x allows us to assume $a = 1$.

As before, we have $\bar{x} = x \otimes 1$, $\bar{y} = 1 \otimes y$, and $\bar{z} = 1 \otimes z$ as possible algebra generators, but the commutators are now $[\bar{x}, \bar{y}] = [\bar{x}, \bar{z}] = 0$ and $[\bar{y}, \bar{z}] = \bar{x}$. Thus $\{\bar{x}^i \bar{y}^j \bar{z}^k\}$ spans C. The coproduct calculation shows that $\bar{x}$ and $\bar{y}$ are primitive and

$$\bar{\psi}(\bar{z}) = \sum_{\substack{r+s=p \\ r,s>0}} \frac{1}{r!s!} \bar{y}^r \otimes \bar{y}^s,$$

which in turn implies that $\bar{x}$ and $\bar{y}$ have height p and that $\bar{z}^p$ is in the span of $\{\bar{x}^i \bar{y}^j \bar{z}^k\}$ for $0 \le i, j, k < p$. Degree considerations and the fact that $\bar{y}$ commutes with $\bar{z}^p$ show that $\bar{z}^p = a\bar{x}^{p-1}\bar{y}$. To determine a, compare terms of the form $- \otimes \bar{y}$ in the coproduct of either side. A long but elementary computation with commutators and binomial coefficients gives $a = 1$. Thus we have the Hopf algebra of type (b).

Note that the Hopf algebra of type (b) is essentially the sub-Hopf algebra of the Steenrod algebra generated by $\bar{y} = P^1$ and $\bar{z} = P^p$.

3.4. Fields which are not perfect

The assumption that k be perfect in the statement of Theorems 3.2 and 3.3 is necessary in order to use Borel's theorem to classify the possible algebra structures on these Hopf algebras. The theory of Section 2 can also be used for this purpose. When this is done we see that Borel's theorem gives a complete list for total rank two, even if k is not perfect. Thus Theorem 3.2 is true for general k. For total rank three the theory yields one additional algebra structure when k is not perfect. However, there is only one possible coproduct for this algebra structure, so Theorem 3.3 is true for general k if one more type of Hopf algebra is added to the list. That is the primitively generated Hopf algebra given by

$$k[x, y]/(y^{p^2}, x^p - ly^p),$$

where l is nonzero in k. If k is perfect, the substitution $x' = y - ax$ with $l = a^p$ shows that this is of type (g-1). For more general fields, some of these Hopf algebras may coincide for certain values of l.

Finally, Theorem 3.9 remains true when k is not perfect since the assumption is only made in order to cite Theorem 3.2.

Note added in proof. The algebra of type g-3 in Theorem 3.3 is not a Hopf algebra and should therefore be removed from the list.

References

[1] J.W. Milnor and J.C. Moore, On the structure of Hopf algebras, Ann. of Math. (2) 81 (1965) 211–264.
[2] W.M. Singer, Extension theory for connected Hopf algebras, J. Algebra 21 (1972) 1–16.
[3] C.W. Wilkerson, The cohomology algebras of finite dimensional Hopf algebras, Trans. Amer. Math. Soc. 264 (1981) 137–150.